CLOSED IDEALS IN THE GROUP ALGEBRA $L^1(G) \cap L^2(G)$ (1)

BY
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0. **Introduction.** In the following, G will denote a locally compact abelian topological group with character group $\hat{G}$. For $1 \leqq p < \infty$, $L^p(G)$ is the Banach space of all complex-valued functions whose pth powers are Haar integrable over G. ($L^p(G)$ is often written L^p when the group G is obvious from the context.) The linear space $L^1(G) \cap L^2(G)$ (denoted $L^1 \cap L^2$) is normed in such a way that, under convolution as multiplication, it is a commutative Banach algebra (§2). It is also proved in §2 that it is regular, semi-simple and that its regular maximal ideal space is $\hat{G}$. It is shown (§3) that the abstract Šilov theorem [8, p. 86] holds for $L^1 \cap L^2$. The standard proof of this theorem in $L^1(G)$ seems to depend upon the uniform boundedness of the approximate identity. A novel aspect of the $L^1 \cap L^2$ case is that a similar proof is obtained despite the fact that every approximate identity in $L^1 \cap L^2$ is unbounded.

An important but unsolved problem of harmonic analysis is the classification of the closed ideals in $L^1(G)$. Using the additional structure supplied by $L^1 \cap L^2$ it is to be expected that more precise results can be obtained about the closed ideals in $L^1 \cap L^2$. If G and $\hat{G}$ are both locally compact metric abelian groups, examples of the more precise results that can be obtained are: (a) If I is a closed proper ideal in $L^1 \cap L^2$, then there exists an $x \in I$ such that the hull of x and the hull of I coincide except for a set of measure zero (Theorem 7.2). (b) For every closed invariant proper subspace $N \subset L^2(G)$, $N \cap L^1 = k(h(N \cap L^1))$ (Corollary 2 of Theorem 7.4). This permits a new characterization of the kernel of E for a class of perfect sets $E \subset \hat{G}$. (A. Denjoy terms these sets "épais en lui-même" in *Leçons sur le calcul des coefficients d'une série trigonométrique*, Paris, 1941, 2ième Partie, p. 100.) (c) The set $\mathscr{I}$ of all closed proper ideals in $L^1 \cap L^2$ which are

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not dense in L^2 is characterized as the set of all closed proper ideals I such that the hull of I has positive measure.

In §6 it is shown (still under the assumption that G is locally compact metric abelian) that if $E \subset \hat{G}$ is a closed set, then E is the hull of a principal ideal in $L^1 \cap L^2$ iff E is a G_δ. The theorem holds under rather more general circumstances(2); in particular, it holds in $L^1(G)$. It follows (from 6.2 and 6.3) that if G and $\hat{G}$ are both locally compact metric abelian, then a hull E for which spectral synthesis holds (if I is any closed ideal having hull E, then I is the kernel of E), must be a closed G_δ set. Consequently, the only instances of Helson's Theorem(3) [4] are given by principal ideals.

1. **Preliminaries and notation.** The following two theorems are useful in the sequel:

THEOREM 1.1. *If G is a locally compact group, then G is normal, and the family of compact neighborhoods of the identity is a basis for the neighborhood system of G at the identity* (*Kelley* [7, 5Y, 5.32 *and* 5.17]).

THEOREM 1.2. *If G is a locally compact abelian group whose character group is $\hat{G}$, then the following are equivalent*;

(a) *G is metrizable*;

(b) *The neighborhood system for the identity $e \in G$ has a countable basis*;

(c) *$\hat{G}$ is σ-compact.*

Proof. That (a) and (b) are equivalent is proved in Kelley [7, p. 186]. That (b) and (c) are equivalent is proved in Hewitt and Ross [5, p. 397] (actually a more general result is proved).

The notations and definitions in this work are, in general, those of Loomis [8]. In particular, $L(E)$ will denote the set of all continuous functions having compact support in E. Also, if A is any set of functions, then A^+ will denote the set of non-negative functions in A.

2. **The Banach algebra** $L^1 \cap L^2$. Let $L^1 \cap L^2$ denote the linear space $L^1(G) \cap L^2(G)$ and observe that the function defined by the equation $\| x \| = \| x \|_1 + \| x \|_2$ for each $x \in L^1 \cap L^2$ is a norm. If multiplication is defined by convolution. it follows that $L^1 \cap L^2$ is a commutative Banach algebra. The conjugate space of $L^1 \cap L^2$ is also obtained in this section. The ideal, S, of $L^1 \cap L^2$-functions whose Fourier transforms have compact support is shown to be dense in $L^1 \cap L^2$ and the regular maximal ideal space of $L^1 \cap L^2$ is found to be $\hat{G}$.

(2) What is required is that A should be a commutative regular semi-simple Banach algebra whose regular maximal ideal space is σ-compact.

(3) If I is a closed proper ideal in $L^1(G)$ such that the boundary of the hull of I contains no nonempty perfect subset, then $I = k(h(I))$.

DEFINITIONS 2.1. Let μ denote Haar measure on G. The set $E \subset G$ is locally null if for every compact set $C \subset G$, $\mu(E \cap C) = 0$. If x, y are μ-measurable functions defined on G such that the set $\{s \in G \mid x(s) \neq y(s)\}$ is locally null, then $x = y$ l.a.e. (locally almost everywhere). Let LL^{∞} denote the equivalence classes of functions which are l.a.e. equal.

THEOREM 2.2. *The linear space $L^1 \cap L^2$ is a Banach space.*

Proof. We have only to show that $L^1 \cap L^2$ is complete. Suppose $\{x_n\} \subset L^1 \cap L^2$ is an $L^1 \cap L^2$-Cauchy sequence, so that there exist $x \in L^1$ and $y \in L^2$ for which $\| x_n - x \|_1 \to 0$ and $\| x_n - y \|_2 \to 0$. Hence there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that $y_n \to x$ a.e., and a subsequence of it, $\{z_n\}$, such that $z_n \to y$ a.e. Thus $z_n \to x$ a.e., so that $x = y$ a.e.

LEMMA 2.3. *Let T be a linear functional on $L^1 \cap L^2$ defined by the equation $T(x) = \int x(t) \cdot \overline{y(t) + z(t)}\, dt$ for each $x \in L^1 \cap L^2$, where $y \in L^2$ and $z \in L^{\infty}$. Then T is bounded, and $\| T \| \leqq \max(\| y \|_2, \| z \|_{\infty})$.*

Proof. Let $M = \max(\| y \|_2, \| z \|_{\infty})$. Then, by Hölder's Inequality, $| T(x) | \leqq \| x \|_2 \ \| y \|_2 + \| x \|_1 \ \| z \|_{\infty} \leqq \| x \| M$. Hence $\| T \| = \sup_{\|x\|=1} | T(x) | \leqq M$.

THEOREM 2.4. *The conjugate space of $L^1 \cap L^2$ is $(LL^{\infty} \times L^2)/Z$, where $Z = \{(g, h) \in LL^{\infty} \times L^2 \mid g + h = 0 \ \text{l.a.e.}\}$.*

Proof. Let ρ be defined on $L^1 \times L^2$ by the equation $\rho(x, y) = \| x \|_1 + \| y \|_2$, and r be defined on $LL^{\infty} \times L^2$ by the equation $r(g, h) = \max(\| g \|_{\infty}, \| h \|_2)$. If $z \in L^p(G)$ and $w \in L^q(G)$, where $1 \leqq p \leqq \infty$, and $1/p + 1/q = 1$, let $\langle z, w \rangle = \int z(s) \cdot \overline{w(s)}\, ds$. If $L^1 \times L^2$ is equipped with the norm ρ, it becomes a Banach space whose conjugate is the Banach space $LL^{\infty} \times L^2$ equipped with the norm r (Schatten [13]). Identify $L^1 \cap L^2$ with $\Delta = \{(x, x) \mid x \in L^1 \cap L^2\}$, which is a closed linear subspace of $L^1 \times L^2$. Let $f \in (L^1 \cap L^2)^*$ (= Banach space conjugate of $L^1 \cap L^2$), and let ϕ be defined on Δ by the equation $\phi(x, x) = f(x)$. By the Hahn-Banach theorem ϕ may be extended without change of norm from the closed linear subspace Δ to all of $L^1 \times L^2$, i.e. ϕ may be extended to a bounded linear functional $F \in LL^{\infty} \times L^2$. Let $F = (F_1, F_2)$, and observe that, if $(x, y) \in L^1 \times L^2$, $F(x, y) = \langle x, F_1 \rangle + \langle y, F_2 \rangle$. If $(x, x) \in \Delta$ it follows that $F(x, x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle = \langle x, F_1 + F_2 \rangle$. It is clear that if $F_1 + F_2 = 0$ l.a.e., $F \equiv 0$ on Δ, hence it follows that we must identify elements of Δ^* (= Banach space conjugate of Δ) differing by an element of Z, where $Z = \{(g, h) \in LL^{\infty} \times L^2 \mid g + h = 0 \ \text{l.a.e.}\}$. Let $\{(g_n, h_n)\} \subset Z$, and suppose that $(g_n, h_n) \to (g, h)$. Then $\| g_n - g \|_{\infty} \to 0$ and $\| h_n - h \|_2 \to 0$, and $g_n + h_n = 0$ l.a.e. for $n = 1,2,3,\cdots$; but, since $\| h_n - h \|_2 \to 0$, there exists a subsequence $\{(p_n, k_n)\} \subset \{(g_n, h_n)\}$ such that $\| k_n - h \|_{\infty} \to 0$, and for each $n = 1,2,3,\cdots$, $p_n + k_n = 0$ l.a.e. Therefore,

$$\| g + h \|_{\infty} = \| p_n + k_n - g - h \|_{\infty} \leqq \| p_n - g \|_{\infty} + \| k_n - h \|_{\infty} \to 0.$$

Hence $g + h = 0$ l.a.e.; i.e., $(g, h) \in Z$, so that Z is closed. Therefore $(LL^\infty \times L^2)/Z$ is a Banach space, and $\phi \in (LL^\infty \times L^2)/Z$; i.e., $(L^1 \cap L^2)^* \subset (LL^\infty \times L^2)/Z$. Now, let $H \in (LL^\infty \times L^2)/Z$. We can find $F \in LL^\infty \times L^2$ such that F belongs to the coset H of $LL^\infty \times L^2$, and for this F, let $f(x) = F(x, x)$. Then $F(x, x) = \langle x, F_1 \rangle + \langle x, F_2 \rangle$, so that

$$|f(x)| \leqq \| x \|_1 \, \| F_1 \|_\infty + \| x \|_2 \, \| F_2 \|_2 \leqq \| x \| \, \| F \|;$$

i.e., $f \in (L^1 \cap L^2)^*$. Therefore $(LL^\infty \times L^2)/Z = (L^1 \cap L^2)^*$.

LEMMA 2.5. *Let $x, y \in L^1 \cap L^2$. Then* $\| x * y \| \leqq \min(\| x \|_1 \, \| y \|, \| x \| \, \| y \|_1)$.

COROLLARY 1. $\| x * y \| \leqq \| x \| \cdot \| y \|$.

COROLLARY 2. *If $x \neq 0$ and $y \neq 0$,* $\| x * y \| < \| x \| \cdot \| y \|$.

COROLLARY 3. *$L^1 \cap L^2$ is a Commutative Banach Algebra.*

DEFINITION 2.6. Let A be a Banach algebra and let P be a directed set. Then the net $\{v_p \in A \mid p \in P\}$ is an approximate identity for A if $\lim_p v_p x = \lim_p x v_p = x$ for each $x \in A$.

Note that it is not required that $\{\| v_p \|\}$ should be bounded. In fact this cannot be required, in general: $L^1 \cap L^r$ with norm $\| \cdot \|_1 + \| \cdot \|_r$ is a Banach algebra if its multiplication is convolution. If G is neither compact nor discrete (of course G is still assumed to be locally compact abelian), $L^1 \cap L^r$ has an approximate identity in the above sense, but it can be shown(4) that any approximate identity in the above sense must be unbounded if $1 < r \leqq 2$.

Let $\mathscr{V}$ denote the family of all precompact (closure is compact) neighborhoods of e, the identity of the group G. Partially order $\mathscr{V}$ by inclusion and designate it by $\{V_p\}$. Then $\{V_p\}$ is a directed set, and we may define a net $\{v_p\}$ of functions on it, by choosing, for each V_p, $v_p \in L^+(V_p)$ such that $\int v_p(s) ds = 1$.

THEOREM 2.7. *The net $\{v_p\}$ defined above is an approximate identity for $L^1 \cap L^2$ (cf. Loomis [8, p. 124]).*

THEOREM 2.8. *Let Σ be a closed subset of $L^1 \cap L^2$. Then Σ is an ideal iff it is a translation-invariant subspace of $L^1 \cap L^2$ (cf. Loomis [8, p. 125]).*

LEMMA 2.9. *Let $v \in L^{1+}$, $\int v(t) dt = 1$ and $\varepsilon > 0$ be given. Then there exists $q \in (L^1 \cap L^2)^+$ such that $\hat{q} \in L(\hat{G})$, $\int q(t) dt = 1$, and $\| q - v \|_1 < \varepsilon$ (Edwards [2, pp. 165–166]).*

THEOREM 2.10. *The ideal $S = \{x \in L^1 \cap L^2 \mid x \in L(\hat{G})\}$ is dense in $L^1 \cap L^2$.*

Proof. Let $x \in L^1 \cap L^2$ and $\varepsilon > 0$ be given. Assume that $x \neq 0$, since $0 \in S$. Choose v from an approximate identity so that $v \in L^{1+}$, and $\|x * v - x\| < \varepsilon/2$. Then by Lemma 2.9, choose $q \in S$ so that $\| q - v \|_1 < \varepsilon / 2 \| x \|$. Hence

(4) By application of the Hausdorff-Young Inequality.

$\| q*x - x\| \leqq \| x*(q - v) \| + \| x*v - x \| < \varepsilon$. Thus $\| q*x - x\| \leqq \varepsilon$ and $(q*x)\hat{} = \hat{q}\hat{x} \in L(\hat{G})$.

THEOREM 2.11. *Let $K \subset \hat{G}$ be any compact set containing $\hat{e}$, and let U be an open neighborhood of K. Then there exists a function $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 1$ on K, $\hat{x} \equiv 0$ off U, and $0 \leqq \hat{x} \leqq 1$.*

Proof. Let V be a symmetric compact neighborhood of $\hat{e}$ sufficiently small that $VVK \subset U$. Set $\Sigma = VK$. Then Σ is compact. Let y, z be the characteristic functions of Σ, V respectively. Since V, Σ are both compact, each has finite measure, so that $y, z \in (L^1 \cap L^2)(\hat{G})$ and therefore $y*z \in (L^1 \cap L^2)(\hat{G})$. Let $\tilde{y}, \tilde{z}, (y*z)\tilde{}$ be the inverse Fourier transforms of y, z and $y*z$ respectively. Then $(y*z)\tilde{} = \tilde{y} \cdot \tilde{z}$. Let $u = (y*z)\tilde{} = \tilde{y} \cdot \tilde{z}$. Then $u \in L^2(G)$, since $y*z \in L^2(\hat{G})$. Also $u \in L^1(G)$ since $\tilde{y}, \tilde{z} \in L^2(G)$. Thus $u \in L^1 \cap L^2(G)$, and $\hat{u}(\alpha) = (y*z)(\alpha)$ a.e., and since each of $\hat{u}$ and $y*z$ is continuous, $\hat{u} = y*z$. Let $x = u/m(V)$. This x is the desired function.

REMARK. By translation, it follows that if K is a compact subset of $\hat{G}$ and if U is any open neighborhood of K, there exists a function $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 1$ on K, $\hat{x} \equiv 0$ off U, and $0 \leqq \hat{x} \leqq 1$.

NOTATION 2.12. Let A be a commutative Banach algebra, and let $\Delta(A)$ denote the set of all continuous homomorphisms of A onto the complex numbers. If A^* is the conjugate space of A, then $\Delta(A)$ is a subset of A^*, and $\Delta(A)$ is locally compact in the weak*-topology of A^*.

Let $\mathfrak{M}$ be the set of all regular maximal ideals of A. Then $\Delta(A)$ may be identified with $\mathfrak{M}$ by associating with $\phi \in \Delta(A)$ the corresponding regular maximal ideal $M_\phi \in \mathfrak{M}_\phi$: $M_\phi \equiv \phi^{-1}(0)$. If I is an ideal of A, the hull $h(I)$ of I is the set of all regular maximal ideals containing I. If $S \subset \mathfrak{M}$, the kernel $k(S)$ of S is the ideal which is the intersection of all the regular maximal ideals $M \in S$. If the closure of a set $S \subset \mathfrak{M}$ is defined as $h(k(S))$, the hull-kernel topology is induced on $\mathfrak{M}$. If this topology coincides with the weak*-topology on $\mathfrak{M}$, A is said to be regular.

Let A be the algebra $L^1 \cap L^2$ and let $M \in \mathfrak{M}$. Then M is $\phi^{-1}(0)$ for some homomorphism $\phi \in \Delta(L^1 \cap L^2)$. Denote $\phi(x)$ by $x'(M)$ for each $x \in L^1 \cap L^2$. Thus x' is a function defined on $\mathfrak{M}$. Fix $M \in \mathfrak{M}$, and let $\alpha_M(s) = x'_s(M)/x'(M)$, where $x \in L^1 \cap L^2$ is so chosen that $x'(M) \neq 0$ (i.e. $x \notin M$).

THEOREM 2.13.

(i) *α_M is a character of G,*

(ii) *α_M is continuous on $G \times \mathfrak{M}$,*

(iii) *If u runs through an approximate identity, $u'_s(M)$ converges uniformly to $\alpha_M(s)$.*

The proof is the same as that in Loomis [8, pp. 135–136].

THEOREM 2.14. *The mapping $M \to \alpha_M$ is a one-to-one mapping of $\mathfrak{M}$ onto the set of all characters of G, and $x'(M) = \int x(s) \cdot \overline{\alpha_M(s)}\, ds$ (cf. Loomis [8, p. 136]).*

THEOREM 2.15. *The topology of $\hat{G}$ (the weak*-topology of $(L^1 \cap L^2)^*$ induced on $\hat{G}$) is the usual topology of $\hat{G}$ as the dual of G.*

Proof. Let F be the set of Fourier transforms of all functions in $L^1 \cap L^2$. Then $F \subset C_0(\hat{G})$, the set of continuous functions vanishing at infinity on $\hat{G}$. Suppose $\alpha, \beta \in \hat{G}$ and $\alpha \neq \beta$. Since $\hat{G}$ is normal, there exist open disjoint neighborhoods $U(\alpha)$, $V(\beta)$, and disjoint compact neighborhoods $K(\alpha) \subset U$ and $C(\beta) \subset V$. By Theorem 2.11 there exist $x, y \in L^1 \cap L^2$ such that $\hat{x} \equiv 1$ on K, $\hat{x} \equiv 0$ on C and $\hat{y} \equiv 1$ on C, $\hat{y} \equiv 0$ on K. Therefore, for each $\alpha \in \hat{G}$ there exists $\hat{x} \in F$ such that $\hat{x}(\alpha) = 1$, and also, F separates the points of $\hat{G}$. By a theorem (5G) of Loomis [8], it follows that the weak topology induced on $\hat{G}$ by F is precisely the one in which the functions of F are continuous; i.e., it is the the usual topology for $\hat{G}$ as the dual of G.

THEOREM 2.16. *$L^1 \cap L^2$ is semi-simple and regular.*

Proof. We have established that if $x \in L^1 \cap L^2$, $x' = \hat{x}$. It follows that if $x' \equiv 0$, $x \equiv 0$ a.e.; i.e., $L^1 \cap L^2$ is semi-simple. By a theorem in Loomis [8, p. 57] to prove that $L^1 \cap L^2$ is regular we have only to prove that if $F \subset \mathfrak{M}$ is closed in the hull-kernel topology, and $\alpha \notin F$, then there exists $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 0$ on F and $\hat{x}(\alpha) \neq 0$. Let $U = \hat{G} - F$ so that U is open and $\alpha \in U$. Choose a compact neighborhood K of α such that $K \subset U$. Apply Theorem 2.11 to obtain $x \in L^1 \cap L^2$ such that $\hat{x} \equiv 0$ off U, $\hat{x} \equiv 1$ on K so that $x \equiv 0$ on F, and $\hat{x}(\alpha) = 1 \neq 0$.

3. **Šilov's Theorem for $L^1 \cap L^2$.** Let A be a commutative Banach algebra. Then A satisfies the condition D if, given $x \in M \in \mathfrak{M}$ there exists a sequence $\{x_n\} \subset A$ such that $\hat{x}_n \equiv 0$ in a neighborhood V_n of M for $n = 1,2,3, \cdots$, and $\| xx_n - x \| \to 0$. If $\mathfrak{M}$ is not compact the condition must also be satisfied for the point at infinity; i.e., for each point $x \in A$, there exists a sequence $\{x_n\} \subset A$ such that $\{\hat{x}_n\} \subset L(\mathscr{M})$, and $\lim_{n \to \infty} \| xx_n - x \| = 0$

ŠILOV'S THEOREM 3.1 (LOOMIS [8, P. 86]). *Let A be a regular semisimple commutative Banach algebra satisfying condition D, and let I be a closed ideal in A. Then I contains every element $x \in k(h(I))$ such that [bd $h(x)$] $\cap\, h(I)$ includes no nonempty perfect set; i.e., is scattered (a closed scattered set is one which contains no nonempty perfect subset).*

Since we have already established that $L^1 \cap L^2$ is regular, commutative, and semi-simple, we have only to show that $L^1 \cap L^2$ satisfies the condition D. We shall first prove that $L^1 \cap L^2$ satisfies the condition D at infinity. After that, the remaining part of this section will be devoted to showing that $L^1 \cap L^2$ satisfies the condition D at finite points. The proof given in Loomis [8, p. 151] that $L^1(G)$ satisfies Ditkin's Condition at finite points appears to depend upon the uniform boundedness of the approximate identity. Since this boundedness is never available in the $L^1 \cap L^2$ case (cf. 2.6), it is somewhat surprising that in spite of this lack a proof similar to the $L^1(G)$ case can be constructed.

LEMMA 3.2. *$L^1 \cap L^2$ satisfies the condition D at infinity.*

Proof. Assume that $\hat{G}$ is not compact. Let $x \in L^1 \cap L^2$, and $\varepsilon > 0$ be given. Use the construction of Theorem 2.10 to obtain $q \in S$ (so that $\hat{q} \in L(\hat{G})$) such that $\| q * x - x \| < \varepsilon$. Clearly then, there exists a sequence $\{x_n\} \subset S$ for which $\lim_n \| x * x_n - x \| = 0$. Since this can be done for every $x \in L^1 \cap L^2$, $L^1 \cap L^2$ satisfies the condition D at infinity.

Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ denote the family of all symmetric Baire neighborhoods of $\hat{e}$ of measure less than or equal to one. Then $\mathscr{U}$ is a directed system under inclusion. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ denote any net of symmetric Baire neighborhoods of $\hat{e}$ defined on $\mathscr{U}$, and satisfying the following conditions:

(i) If $V \in \{V_\lambda\}$, $\bar{V}$ is compact;

(ii) Given $U_\lambda \in \mathscr{U}$, $\bar{V}_\lambda \subset U_\lambda$ and $m(U_\lambda) < 4m(V_\lambda)$ (where m is Haar measure on $\hat{G}$);

(iii) Given $U_\lambda \in \mathscr{U}$ and V_λ, there exists a neighborhood $W(\hat{e})$ depending on U_λ and V_λ, such that $V_\lambda W \subset U_\lambda$.

LEMMA 3.3. *There exists a net $\{z_\lambda\} \subset L^1 \cap L^2$ defined on $\mathscr{U}$ such that for every $\lambda \in \Lambda$:*

(i) $\| z_\lambda \| < 3$, *and*

(ii) $\hat{z}_\lambda \equiv 1$ *on some neighborhood of* $\hat{e}$.

Proof. Given U_λ, let V_λ be the corresponding set in the net of sets defined above. Let $\hat{u}_\lambda$, $\hat{v}_\lambda$ be the characteristic functions of U_λ, V_λ respectively, and let $z_\lambda = \hat{u}_\lambda * \hat{v}_\lambda / m(V_\lambda)$. Since $\hat{u}_\lambda$, $\hat{v}_\lambda$ and $\hat{z}_\lambda$ all belong to $L^1 \cap L^2(\hat{G})$, the inverse Fourier-Plancherel transform of each exists. These may be designated as u_λ, v_λ and $\hat{z}_\lambda$, respectively.

Proof of (i). $\|z_\lambda\|_2 = \|\hat{z}_\lambda\|_2 = [1/m(V_\lambda)] \cdot \|\hat{u}_\lambda * \hat{v}_\lambda\|_2 \leqq [1/m(V_\lambda)] \cdot \| \hat{u}_\lambda \|_2 \cdot \| \hat{v}_\lambda \|_1 \leqq 1$. Thus $\| z_\lambda \|_2 \leqq 1$. Similarly $\| z_\lambda \|_1 = [1/m(V_\lambda)] \cdot \|u_\lambda v_\lambda \|_1 < 2$. Hence $\| z_\lambda \| < 3$.

Proof of (ii). Corresponding to U_λ and V_λ there exists a neighborhood $W = W(\hat{e})$ such that $V_\lambda W \subset U_\lambda$. Let $\beta \in W$. Then

$$\hat{z}_\lambda(\beta) = [1/m(V_\lambda)] \cdot (\hat{u}_\lambda * \hat{v}_\lambda)(\beta) = [1/m(V_\lambda)] \int_{V_\lambda} \hat{u}_\lambda(\alpha\beta)\, d\alpha = 1.$$

Hence $\hat{z}_\lambda \equiv 1$ on $W(\hat{e})$.

Let $C \subset G$ be a compact subset, and let $\varepsilon > 0$ be given. Then $U(C, \varepsilon/5, \hat{e})$ is open in $\hat{G}$, where $U(C, \varepsilon/5, \hat{e}) = \{\alpha \in \hat{G} \mid |1 - (s, \alpha)| < \varepsilon/5, \text{ all } s \in C\}$. Since $\mathscr{U}$ contains a basis for the topology of $\hat{G}$ at $\hat{e}$, there exists a λ_0 such that if $\lambda < \lambda_0$, $U_\lambda^2 \subset U(C, \varepsilon/5, \hat{e})$. For convenience of notation, let S_λ denote the net $\{z_\lambda\}$ constructed in Lemma 3.3, and let $S_{\lambda_0} = \{z_\lambda \mid \lambda > \lambda_0\}$.

LEMMA 3.4. *Given $\varepsilon > 0$, there exists λ_0 such that if $z \in S_{\lambda_0}$, then*

$$\| z - z_s \| < \varepsilon$$

for every $s \in C$.

Proof. Choose an appropriate λ_0 as above so that if $\lambda > \lambda_0$, $U_\lambda^2 \subset U(C,\varepsilon/5,\hat{e})$. Let $z \in S_{\lambda_0}$ and suppose that $z = uv/m(V)$. Let $s \in C$, and note that $\hat{z} \equiv 0$ off UV. Then $\| z - z_s \|_2^2 = \| \hat{z} - \hat{z}_s \|_2^2 = \int_{UV} |\hat{z}(\alpha)|^2 \cdot |1 - (s,\alpha)|^2 \, d\alpha < (\varepsilon/5)^2$. Hence $\| z - z_s \|_2 < \varepsilon/5$. Similarly $\| u - u_s \|_2 < [m(U)]^{1/2}(\varepsilon/5)$, and

$$\| v - v_s \|_2 < [m(V)]^{1/2}(\varepsilon/5).$$

We observe that

$$\begin{aligned} \| z - z_s \|_1 &\leqq [1/m(V)]\,[\| u(v - v_s) \|_1 + \| v_s(u - u_s) \|_1] \\ &< 2(\varepsilon/5)\left[\frac{m(U)}{m(V)}\right]^{1/2} < 4\varepsilon/5, \end{aligned}$$

so that $\| z - z_s \| < \varepsilon$.

COROLLARY. *If $x \in L^1 \cap L^2$, and $\hat{x}(\hat{e}) = 0$, then $\lim_\lambda \| x * z_\lambda \| = 0$.*

Proof. Let $\delta > 0$ be given, and choose $C \subset G$ to be compact, symmetric and such that $\int_{G-C} |x(s)| \, ds < \delta/12$. Set $\varepsilon = \delta/2\| x \|_1$, and choose λ_0 as before so that if $z \in S_{\lambda_0}$, then $\| z - z_s \| < \varepsilon$ for every $s \in C$. Hence

$$(x * z)(t) = \int x(s)\,[z(ts^{-1}) - z(t)]\,ds.$$

We observe that $\| z * x \|_1 = \sup_{\|h\|_\infty = 1} | \langle z * x, h \rangle |$, and that

$$\| z * x \|_2 = \sup_{\|p\|_2 = 1} | \langle z * x, p \rangle |.$$

Thus, by a straightforward computation,

$$(*) \qquad \| x * z \| \leqq \int_C | x(s) | \, \| z_{s^{-1}} - z \| \, ds + \int_{G-C} | x(s) | \, \| z_{s^{-1}} - z \| \, ds.$$

If $z \in S_{\lambda_0}$, then $s \in C$ implies that $\| z_{s^{-1}} - z \| < \varepsilon$. In this case the inequality $(*)$ becomes $\| x * z \| < \varepsilon \| x \|_1 + 2\| z \| \delta/12 < \delta$, so that $\lim_\lambda \| x * z_\lambda \| = 0$.

THEOREM 3.5. *There exists a net $\{v_q\} \subset L^1 \cap L^2$ such that each $\hat{v}_q = 0$ in a neighborhood of $\hat{e}$ (depending on v_q) and such that if $x \in L^1 \cap L^2$, and $\hat{x}(\hat{e}) = 0$, then $\lim_q \| x * v_q - x \| = 0$.*

Proof. Let $\{u_p\}$ be the approximate identity defined in Theorem 2.7 and let $\{z_\lambda\}$ denote the net defined in Lemma 3.3. Let $v(p,\lambda) = u_p - z_\lambda u_p$. Clearly $v(p,\lambda) \in L^1 \cap L^2$. The set of all ordered pairs (p,λ) may be directed by: (p_1, λ_1)

$> (p_2, \lambda_2)$ iff $p_1 > p_2$ and $\lambda_1 > \lambda_2$. If we allow q to run through this directed set, $\{v_q\}$ is a net, and we note that $\hat{v}(p, \lambda) = \hat{u}_p(1 - \hat{z}_\lambda) \equiv 0$ in the neighborhood of $\hat{e}$ where $\hat{z}_\lambda \equiv 1$. Finally $\| v(p, \lambda) * x - x \| \leqq \| u_p * x - x \| + \| u_p \|_1 \| x * z_\lambda \|$. Hence $\lim_q \| v_q * x - x \| \leqq \lim_p \| u_p * x - x \| + \lim_\lambda \| u_p \|_1 \| x * z_\lambda \| = 0$.

COROLLARY 1. *$L^1 \cap L^2$ satisfies condition D.*

Proof. In the above theorem we have just established that $L^1 \cap L^2$ satisfies condition D at $\hat{e}$. The condition D follows for all other finite points upon translation. It was established for the point at infinity in Lemma 3.2.

COROLLARY 2. *Šilov's theorem is valid for $L^1 \cap L^2$.*

4. **Translation-invariant subspaces of $L^2(G)$.** Let the notation $[m]$ following an assertion denote that the assertion is valid except for sets of zero m-measure (on $\hat{G}$).

EXAMPLE. $E \subset F\ [m]$ means that $m(E - F) = 0$.

If $x \in L^2(G)$, $N(x)$ will denote the set of all finite linear combinations of translates of x. $H(x)$ will denote the L^2-closure of $N(x)$. The spaces $L^1(G)$, $L^2(G)$ will be written as L^1, L^2 respectively unless possible ambiguity prevents this. If N is any subspace of L^2, N is invariant if, for every $s \in G$, $x \in N$ implies that $x_s \in N$. If $x \in L^2$, define $h(x) = \{\alpha \in \hat{G} \mid \hat{x}(\alpha) = 0\}\ [m]$.

The result (4.1) of this section is taken from S. Bochner and K. Chandrasekharan [1, pp. 148–149], where it is established for the case of $G = R$. Their proof carries over without change to the general case, so there is no need to reproduce it here.

THEOREM 4.1. *Let $x, y \in L^2$. Then $x \in H(y)$ iff $h(y) \subset h(x)\ [m]$.*

In this section, let N denote an arbitrary closed *proper* ($\{0\} \neq N$ and $N \neq L^2$) subspace of L^2, invariant under translation. If $E \subset L^2$, cl(E) will denote the L^2-closure of the set E.

LEMMA 4.2. *If $x \in N$, $m(h(x)) > 0$.*

LEMMA 4.3. *Let $x, y \in N$. Then there exists $z \in N$ for which*

$$h(z) = h(x) \cap h(y)[m].$$

THEOREM 4.4. *Let $\{x_n\} \subset N$. Then there exists $x_0 \in N$ such that*

$$h(x_0) = \bigcap_{n=1}^{\infty} h(x_n)\ [m].$$

Proof. Without loss of generality, assume that $\| x_n \|_2 > 0$ for $n = 1,2,3, \cdots$, and let $c_k = \{2^k \cdot \| x_k \|_2\}^{-1}$ for each $k = 1,2,3, \cdots$. Let $\hat{p}_n = \sum_{k=1}^{n} c_k | \hat{x}_k |$, and let p_n be the inverse Fourier-Plancherel transform of $\hat{p}_n$. Then it is clear that $\{p_n\} \subset N$, and $h(p_n) = \bigcap_{k=1}^{n} h(x_k)\ [m]$ by Lemma 4.3. There exists $x_0 \in N$ for which $\lim_n \| p_n - x_0 \|_2 = 0$. Hence $\bigcap_{n=1}^{\infty} h(x_n) = \bigcap_{n=1}^{\infty} h(p_n) \subset h(x_0)\ [m]$, and consequently, we have only to prove that for each n, $h(x_0) \subset h(p_n)$ $[m]$. These remarks lead to a straightforward proof by contradiction.

THEOREM 4.5. *Let E be a measurable subset of $\hat{G}$, and suppose that for some $x' \in N$, $m(E \cap h(x'))$ is finite. Then there exists a $z \in N$ such that for every $x \in N$, $E \cap h(z) \subset E \cap h(x)$ $[m]$.*

Proof. Let $c = \inf_{x \in N} m(E \cap h(x)) < \infty$. Choose a sequence $\{x_n\} \subset N$ such that $m(E \cap h(x_n)) < c + 1/n + 1$ $(n = 1,2,3, \cdots)$. Then, by Theorem 4.4, there exists a $z \in N$ such that $h(z) = \bigcap_{n=1}^{\infty} h(x_n)$ $[m]$. Hence $m(E \cap h(z)) = c$. Let $x \in N$. Then, by Lemma 4.3 there exists $y \in N$ for which $h(y) = h(x) \cap h(z)$ $[m]$. Thus $m(E \cap h(x) \cap h(z)) = c$. Observe that $h(x) \cap h(z)$ and $h(z) - h(x)$ are disjoint and that $h(z) = [h(x) \cap h(z)] \cup [h(z) - h(x)]$. Therefore $c = m(E \cap h(z)) = c + m(E \cap [h(z) - h(x)])$. Thus $m([E \cap h(z)] - [E \cap h(x)]) = 0$.

THEOREM 4.6. *Let G be metric, and let N be a closed proper invariant subspace of $L^2(G)$. Then there exists a $z \in N$ such that $H(z) = N$.*

Proof. Since $\hat{G}$ is σ-compact, we may set $\hat{G} = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact; thus $0 \leqq m(K_n) < \infty$, for $n = 1,2,3, \cdots$. Hence if $x \in N$, $m(K_n \cap h(x))$ is finite, and we apply Theorem 4.5 to obtain a sequence $\{z_n\} \subset N$ such that for every $x \in N$, $K_n \cap h(z_n) \subset K_n \cap h(x)$ $[m]$. By Theorem 4. 4, there exists a $z \in N$ such that $h(z) = \bigcap_{n=1}^{\infty} h(z_n)$ $[m]$. Suppose $x \in N$; then except for a null set,

$$h(z) = \bigcup_{n=1}^{\infty} [K_n \cap h(z)] \subset \bigcup_{n=1}^{\infty} [K_n \cap h(z_n)] \subset \bigcup_{n=1}^{\infty} [K_n \cap h(x)] = h(x).$$

By Theorem 4.1, $x \in H(z)$; i.e., $N \subset H(z)$. Since $z \in N$, $H(z) \subset N$, so that $N = H(z)$.

5. **The closed ideals I and $I^{\perp}$ in $L^1 \cap L^2$.** If I is an ideal in $L^1 \cap L^2$, cl I will be denoted by J. The class of all closed proper ideals I in $L^1 \cap L^2$ for which $J \neq L^2$ will be denoted by $\mathscr{I}$. I is symmetric if $x \in I$ implies that $x^* \in I$, where $x^*(s) = \overline{x(s^{-1})}$ $(s \in G)$. If N is a closed subspace of $L^2(G)$, we shall denote the orthogonal complement of N by $N^{\perp}$. Let $I \in \mathscr{I}$, and let $(x,y) = \int x(s)\overline{y(s)}\, ds$. Then, define $I^{\perp} = \{x \in L^1 \cap L^2 \mid (x, y) = 0 \text{ for all } y \in I\}$.

REMARK 5.1. If M is a regular maximal ideal of $L^1 \cap L^2$, then M is symmetric. Thus, if I is a closed ideal in $L^1 \cap L^2$ such that $I = k(h(I))$, then I is symmetric.

LEMMA 5.2. *Let N be a closed proper invariant subspace of L^2 such that $N \cap L^1 \neq \{0\}$. Then $N \cap L^1 \in \mathscr{I}$ and $N \cap L^1$ is symmetric.*

THEOREM 5.3. *Let $I \in \mathscr{I}$. Then $I^{\perp} = J^{\perp} \cap L^1$.*

Proof. Let $y \in J$. Then there exists a sequence $\{y_n\} \subset I$ such that $\|y_n - y\|_2 \to 0$. If $x \in I^{\perp}$, $(x, y_n) = 0$ $n = 1,2,3, \cdots$, and since strong convergence implies weak convergence, it follows that $(x, y) = 0$. Thus if $x \in I^{\perp}$, $(x, y) = 0$ for every $y \in J$, so that $x \in J^{\perp} \cap L^1$; i.e., $I^{\perp} \subset J^{\perp} \cap L^1$. Now let $x \in J^{\perp} \cap L^1$. Then if $y \in I$, $(x, y) = 0$ since $y \in J$. Thus $x \in I^{\perp}$.

LEMMA 5.4. *Let $I \in \mathscr{I}$. Then J and $J^{\perp}$ are closed proper invariant subspaces of L^2.*

COROLLARY. *If $I \in \mathscr{I}$, $I^{\perp} = \{0\}$,* or *$I^{\perp} \in \mathscr{I}$. In either event, $I^{\perp}$ is symmetric.*

Let the ideal $\underline{I \oplus I^{\perp}}$ be defined as the direct sum of the two ideals I and $I^{\perp}$. The notation $\overline{I \oplus I^{\perp}}$ will denote the $L^1 \cap L^2$-closure of $I \oplus I^{\perp}$.

THEOREM 5.5. *Let I_1 and I_2 be closed proper ideals of $L^1 \cap L^2$ such that $I_1 \cap I_2 = \{0\}$. Then $h(I_1) \cup h(I_2) = \hat{G}$.*

Proof. Let $E = h(I_1) \cup h(I_2)$, and suppose that $E \neq \hat{G}$. Then there exists $\alpha \in \hat{G}$ such that $\alpha \notin E$, and there exist open disjoint neighborhoods $V_1(\alpha)$ and $V_2(E)$. The compact neighborhoods of α are a basis for the topology of $\hat{G}$ at α, so there exists an open neighborhood $U_1(\alpha)$ such that $\bar{U}_1$ is a compact subset of V_1. By Theorem 2.11 there exists $0 \neq y \in L^1 \cap L^2$ such that $\hat{y} \equiv 0$ on $\bar{U}_1$, $\hat{y} \equiv 0$ off V_1, and $0 \leqq \hat{y} \leqq 1$. Observe that $E \subset V_2$ (open) $\subset \hat{G} - V_1 \subset h(y)$ so that $E \subset h(y)^0$ ($=$ interior of $h(y)$). Thus $h(I_1) \subset h(y)^0$ and $h(I_2) \subset h(y)^0$, so that $y \in I_1$ and $y \in I_2$; i.e., $y \in I_1 \cap I_2$, i.e., $y = 0$. This is a contradiction.

COROLLARY. *Let $I \in \mathscr{I}$. Then $h(I) \cup h(I^{\perp}) = \hat{G}$.*

THEOREM 5.6. *If $I \in \mathscr{I}$, then $I^{\perp} \in \mathscr{I}$ iff $h(I)^0 \neq \varnothing$.*

Proof. Part 1. (Necessity) Claim: If $h(I)^0 = \varnothing$, then $I^{\perp} = \{0\}$. In fact, it follows from the above corollary that $h(I) \cup h(I^{\perp}) = \hat{G}$, and $\hat{G} - h(I^{\perp}) \subset h(I)^0 = \varnothing$; so that $\hat{G} - h(I^{\perp}) = \varnothing$, and $h(I^{\perp}) = \hat{G}$; i.e., $I^{\perp} = \{0\}$.

Part 2. (Sufficiency) Claim: If $h(I)^0 \neq \varnothing$, then $I^{\perp} \neq \{0\}$. In fact, let $\alpha \in h(I)^0$, and let $C = C(\alpha)$ be a compact neighborhood of α such that $C \subset h(I)^0$. By Theorem 2.11 there exists $0 \neq y \in L^1 \cap L^2$ such that $\hat{y} \equiv 1$ on C, $\hat{y} \equiv 0$ off $h(I)^0$, and $0 \leqq \hat{y} \leqq 1$. Thus if $x \in I$, and $\beta \in h(I)$, $\hat{x}(\beta) = 0$; and if $\beta \notin h(I)$, $\beta \notin h(I)^0$, so that $\hat{y}(\beta) = 0$. Therefore $\hat{x}\overline{\hat{y}} = \hat{x}\hat{y} \equiv 0$, and therefore $\int \hat{x}(\beta) \cdot \overline{\hat{y}(\beta)}\, d\beta = 0$. Thus $(x, y) = (\hat{x}, \hat{y}) = 0$ for every $x \in I$. Hence $0 \neq y \in I^{\perp}$; i.e., $I^{\perp} \neq \{0\}$.

THEOREM 5.7. *Let I_1 and I_2 be closed proper ideals in $L^1 \cap L^2$ such that $I_1 \cap I_2 = \{0\}$. Then $h(I_1) \cap h(I_2) = h(I_1 \oplus I_2)$.*

COROLLARY $h(\overline{I_1 \oplus I_2}) = h(I_1) \cap h(I_2)$.

6. Closed ideals in $L^1 \cap L^2$ when G is metric. The topological group G has been assumed to be a locally compact abelian group. It will be assumed from this point on that, in addition to this, G is metric. It follows by Theorem 1.2 that $\hat{G}$ is σ-compact. Hence every F_σ set in $\hat{G}$ is also σ-compact.

DEFINITION 6.1. If $x \in L^1 \cap L^2$, the ideal $I(x)$ denotes the closed ideal generated by x together with its translates. The linear subspace $N(x)$ denotes the set of all finite linear combinations of translates of x. Thus the $L^1 \cap L^2$-closure of $N(x)$ is $I(x)$.

LEMMA 6.2. *If* $x \in L^1 \cap L^2$, *then* $h[I(x)] = h(x)$.

Proof. Observe that if $y \in N(x)$, $h(x) \subset h(y)$. Let $z \in I(x)$. Then, since $N(x)$ is $L^1 \cap L^2$-dense in $I(x)$, there exists a sequence $\{z_n\} \subset N(x)$ such that $\|z - z_n\| \to 0$. If $\alpha \in h(x)$, then $\hat{z}_n(\alpha) = 0$ for $n = 1,2,3, \cdots$. Hence

$$|\hat{z}(\alpha)| \leqq \|\hat{z} - \hat{z}_n\|_\infty \leqq \|z - z_n\|,$$

and since $\|z - z_n\| \to 0$, $\hat{z}(\alpha) = 0$. Thus $h(x) \subset h(z)$. Since

$$h[I(x)] = \bigcap \{h(z) \mid z \in I(x)\},$$

it follows that $h(x) \subset h[I(x)]$. But $x \in I(x)$, so that $h[I(x)] \subset h(x)$. Consequently, $h(x) = h[I(x)]$.

THEOREM 6.3. *Let I be a closed ideal of $L^1 \cap L^2$. Then there exists an $x \in I$ such that $h(x) = h(I)$ iff $h(I)$ is a G_δ set.*

Proof. If $h(I) = h(x)$ for some $x \in I$, then since $\hat{x}$ is continuous, $h(I)$ is a G_δ set. Now suppose $h(I)$ is a G_δ set, and let $U = \hat{G} - h(I)$ so that U is an open F_σ-set and $U = \bigcup_{n=1}^{\infty} K_n$, with $K_n \subset U$ and K_n compact. Since $K_n \cap h(I) = \varnothing$, there exist disjoint open neighborhoods $U_n(K_n)$ and $V_n(h(I))$. By Theorem 2.11, there exists $w_n \in L^1 \cap L^2$ such that $\hat{w}_n \equiv 1$ on K_n, $\hat{w}_n \equiv 0$ off U_n, and $0 \leqq \hat{w}_n \leqq 1$. Observe that V_n (open) $\subset h(w_n)$, so that $V_n \subset h(w_n)^0$. Therefore $h(I) \subset h(w_n)^0$, since $h(I) \subset V_n$. Hence $\{w_n\} \subset I$.

Let $x_m = \sum_{k=1}^{m} w_k \cdot \{2^k \| w_k \|\}^{-1}$. Then $\{x_m\} \subset I$, and $\{x_m\}$ is $L^1 \cap L^2$-Cauchy so there exists $x \in L^1 \cap L^2$ such that $x_m \to x$. But I is closed, so it follows that $x \in I$. Therefore $h(I) \subset h(x)$. Let $\alpha \notin h(I)$; i.e., $\alpha \in U$. Then there exists some n for which $\alpha \in K_n$, so that $\hat{w}_n(\alpha) = 1$, and hence $\hat{x}(\alpha) \neq 0$. Therefore $\alpha \notin h(x)$; i.e., $h(x) \subset h(I)$.

COROLLARY. *Let I be a closed ideal in $L^1 \cap L^2$. If $h(I)$ is a G_δ set with a scattered boundary, then there exists an $x \in I$ such that $I = I(x)$.*

Proof. This follows from Šilov's Theorem (Theorem 3.5).

REMARK. This corollary shows that if $\hat{G}$ is also metric, then the only instances of Helson's Theorem [4] for $L^1 \cap L^2$ (and, similarly for $L^1(G)$) are given by principal ideals. Theorem 6.3 shows that if $E \subset \hat{G}$ is closed, then E is the hull of a closed principal ideal in $L^1 \cap L^2$ iff E is a G_δ set. Thus, for example, if G is σ-compact, every closed set in $\hat{G}$ is the hull of a closed principal ideal (since $\hat{G}$ would then be metric), and any nonprincipal closed ideal in $L^1 \cap L^2$ would therefore provide an example of an ideal for which spectral synthesis fails $[I \neq I(x)$, but $h(I(x)) = h(x) = h(I)]$.

It should be remarked here that Schwartz' counterexample to spectral synthesis in $L^1(R^n)$ $(n \geqq 3)$ carries over to $L^1 \cap L^2(R^n)$ $(n \geqq 3)$ with only minor modifications. The proof of the following theorem, therefore, is omitted.

THEOREM. *There exists* $x \in L^1 \cap L^2(R^n)$, *for* $n \geqq 3$, *such that* $x \notin I(x * x)$ (*Reiter* [11, *pp.* 469–470]).

7. The family $\mathscr{I}$ of ideals of $L^1 \cap L^2$.

THEOREM 7.1. *Let* I *be a closed nonzero ideal of* $L^1 \cap L^2$. *If* $m(h(I)) > 0$, *then* $I \in \mathscr{I}$.

Proof. We have only to prove that $J \neq L^2$. We will accomplish this by assuming that $J = L^2$ and showing that this leads to a contradiction.

By Theorem 6.3, there exists a function $u_1 \in L^1 \cap L^2$ such that $\hat{u}_1 > 0$ on $\hat{G}$. Let $y =$ the characteristic function of $h(I)$, and set $\hat{u} = \hat{u}_1 y$. Then $\hat{u} \in L^2(\hat{G})$, so that, by the Plancherel theorem, there exists $u \in L^2(G)$ such that the Fourier transform of u is equal to $\hat{u}$ a.e. Let $\int_{h(I)} |\hat{u}(\alpha)|^2 d\alpha = p^2 > 0$. Since $u \in L^2$, there exists a sequence $\{x_n\} \subset I$ such that $\lim_n \| u - x_n \|_2 = 0$. Hence $0 = \lim_n \| \hat{u} - \hat{x}_n \|_2^2 \geqq p^2 > 0$. This is the desired contradiction.

COROLLARY(5). *If* $\hat{G}$ *has a closed subset* E *of positive measure such that* $E^0 = \varnothing$, *then there exists a closed proper invariant subspace* $N \subset L^2(G)$ *for which* $N \cap L^1 = \{0\}$.

Proof. Let $I = k(E)$, so that $h(I) = E$ and $m(h(I)) > 0$. Then $I \in \mathscr{I}$ by the above theorem and $h(I)^0 = \varnothing$. Hence $I^\perp = \{0\}$ by Theorem 5.6. But $I^\perp = J^\perp \cap L^1$ by Theorem 5.3 and $J^\perp$ is a closed proper invariant subspace of $L^2(G)$ by Theorem 5.4. Therefore the desired subspace is $N = J^\perp$.

EXAMPLE. Let G be the real line under addition, and let E be a Cantor set of positive measure. Note that if G is compact, $\hat{G}$ is discrete so that no set $E \neq \varnothing$ can be found such that $E^0 = \varnothing$. However, in this case, $N \cap L^1 = N$ for every $N \subset L^2(G)$.

THEOREM 7.2. *If* I *is any closed proper ideal in* $L^1 \cap L^2$, *there exists* $x \in I$ *such that* $h(x) = h(I)$ $[m]$.

Proof. Let $\{U_n\}$ be a family of open neighborhoods of $h(I)$ such that $m(U_n - h(I)) < 1/n$ $(n = 1,2,3,\cdots)$. If $F_n = \hat{G} - U_n$, then F_n is closed, and therefore σ-compact. Hence, let $F_n = \bigcup_{k=1}^{\infty} K_{nk}$, where K_{nk} is compact for each

(5) A proof of the fact that every nondiscrete locally compact group contains a compact nowhere dense subset of positive measure was communicated to the author in the summer of 1963 by K. A. Ross and K. Stromberg.

$n, k = 1,2,3,\cdots$. Observe that $h(I) \cap K_{nk} = \varnothing$ so that we may choose disjoint open neighborhoods U_{nk}, V_{nk} of $h(I)$ and K_{nk} respectively, and we can always choose $U_{nk} \subset U_n$. Let $x_{nk} \in L^1 \cap L^2$ be constructed by Theorem 2.11 so that $\hat{x}_{nk} \equiv 1$ on K_{nk}, and $\hat{x}_{nk} \equiv 0$ off V_{nk}. Then $h(I) \subset U_{nk} \subset h(x_{nk})^0$, and it follows that $x_{nk} \in I$. Let $y_{np} = \sum_{k=1}^{p} x_{nk} \cdot \{2^k \| x_{nk} \|\}^{-1}$. Then $\{y_{np}\} \subset I$ and $\{y_{np}\}$ is $L^1 \cap L^2$-Cauchy and I is closed, so that $\lim_p \| y_{np} - x_n \| = 0$ for some $x_n \in I$. By proceeding n this manner, we obtain a sequence $\{x_n\} \subset I$. By construction $\hat{x}_n > 0$ on each K_{nk} (for $k = 1,2,3,\cdots$); i.e., $\hat{x}_n > 0$ on F_n, and $\hat{x}_n \equiv 0$ on each U_{nk} ($k = 1,2,3,\cdots$). Now let $E = \bigcap_{n=1}^{\infty} U_n$, so that $m(E - h(I)) = 0$, and proceeding as before, let $x = \sum_{n=1}^{\infty} x_n \cdot \{2^n \| x_n \|\}^{-1}$. Then $x \in I$, and $\hat{x} > 0$ on each $F_n = \hat{G} - U_n$ (for $n = 1,2,3,\cdots$). Hence $\hat{x} > 0$ on $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (\hat{G} - U_n) = \hat{G} - \bigcap_{n=1}^{\infty} U_n = \hat{G} - E$. Moreover $\hat{x} \equiv 0$ on $h(I)$, so that $h(x) = h(I)$ $[m]$.

Corollary 1. *If I is a closed proper ideal in $L^1 \cap L^2$, then $J = H(x)$.*

Proof. Let $y \in J$. Then there exists a sequence $\{y_n\} \subset I$ such that $\|y - y_n\|_2 \to 0$. Let $F = h(x) - h(y)$ $[m]$, where $x \in I$, and $h(I) = h(x)$ $[m]$ as in the above theorem. Suppose $F \neq \varnothing$ $[m]$, and let $\int_F |\hat{y}(\alpha)|^2 d\alpha = \delta > 0$, since $|\hat{y}(\alpha)|^2 > 0$ a.e. on F. Then $\int_F |\hat{y}(\alpha) - \hat{y}_n(\alpha)|^2 d\alpha = \int_F |\hat{y}(\alpha)|^2 d\alpha = \delta > 0$, since each $\hat{y}_n \equiv 0$ a.e. on F. It follows that $\lim_n \| y - y_n \|_2 \neq 0$, which contradicts our initial assumption. Hence $F = \varnothing$ $[m]$, and if $y \in J$, $h(x) \subset h(y)$ $[m]$. By Theorem 4.1, it follows that $y \in H(x)$. Hence $J \subset H(x)$, so that $J = H(x)$.

The proofs of the following results are direct applications of Theorem 7.1, Theorem 4.1, and the above Corollary 1.

Corollary 2. *Let I be a closed proper ideal in $L^1 \cap L^2$. Then $I \in \mathscr{I}$ iff $m(h(I)) > 0$.*

Corollary 3. *Let I_1, I_2 be closed proper ideals in $L^1 \cap L^2$. Then* cl $I_1 =$ cl I_2 *iff $h(I_1) = h(I_2)$ $[m]$.*

Theorem 7.3. *The group $\hat{G}$ is connected iff for every pair $I, I^{\perp} \in \mathscr{I}$, $\overline{I \oplus I^{\perp}}$ is a proper ideal in $L^1 \cap L^2$.*

Proof. Suppose that $\hat{G}$ is connected and that for some pair $I, I^{\perp} \in \mathscr{I}$, $\overline{I \oplus I^{\perp}}$ is not proper; i.e., $\overline{I \oplus I^{\perp}} = L^1 \cap L^2$. Then by the corollary to Theorem 5.7, $\varnothing = h(I) \cap h(I^{\perp})$ and by the corollary to Theorem 5.5, $h(I) \cup h(I^{\perp}) = \hat{G}$. Hence, $\hat{G}$ is not connected, contrary to our assumption. Conversely, suppose that $\hat{G}$ is not connected; i.e., suppose $\hat{G} = P \cup Q$ where P and Q are open-closed and disjoint in $\hat{G}$. Let $j(P) = \{x \in L^1 \cap L^2 \mid \hat{x} \in L(\hat{G})$ and $P \subset h(x)^0\}$ and let $I = L^1 \cap L^2$-closure of $j(P)$. Thus, $I \in \mathscr{I}$, $I^{\perp} \in \mathscr{I}$ and $h(I) = P$. Let $I' = L^1 \cap L^2$-closure of $j(Q)$, and observe that since Q is open-closed, $x \in I'$ iff $Q \subset h(x)$. Note that $I' \subset I^{\perp}$ and that $Q = \hat{G} - h(I) \subset h(I^{\perp})$. Therefore, if $x \in I^{\perp}$, $Q \subset h(x)$, so that $x \in I'$; i.e., $I^{\perp} = L^1 \cap L^2$-closure of $j(Q)$. Hence

$$h(\overline{I \oplus I^{\perp}}) = h(I) \cap h(I^{\perp}) = P \cap Q = \varnothing,$$

so that $\overline{I \oplus I^{\perp}} = L^1 \cap L^2$.

THEOREM 7.4. *If $I \in \mathscr{I}$, then $k(h(I)) \subset J \cap L^1$.*

Proof. By Theorem 7.2, there exists $x \in I$ such that $h(I) = h(x)$ $[m]$, and by Corollary 1 of the same theorem $H(x) = J$. Let $y \in k(h(I))$, so that $h(I) \subset h(y)$. Then $h(x) \subset h(y)$ $[m]$, and by Theorem 4.1, $y \in H(x) = J$. Since $y \in L^1 \cap L^2$, it follows that $y \in J \cap L^1$. Hence $k(h(I)) \subset J \cap L^1$.

COROLLARY 1. *If $I \in \mathscr{I}$, then $J \cap L^1 = k(h(J \cap L^1))$.*

COROLLARY 2. *If N is a closed proper invariant subspace of $L^2(G)$, then $N \cap L^1 = k(h(N \cap L^1))$. In particular, if $I, I^{\perp} \in \mathscr{I}$, then $I^{\perp} = k(h(I^{\perp}))$.*

Proof. We assume that $N \cap L^1 \neq 0$ without loss of generality. The above Corollary 1 implies that $N \cap L^1 = k(h(L^1 \cap \mathrm{cl}(N \cap L^1)))$. The observation that $N \cap L^1 = L^1 \cap \mathrm{cl}(N \cap L^1)$ completes the proof.

THEOREM 7.5. *Let $I, I^{\perp} \in \mathscr{I}$. If $I \oplus I^{\perp} = k(h(I \oplus I^{\perp}))$, then $I = k(h(I))$.*

Proof. By Theorem 5.7, $h(I \oplus I^{\perp}) = h(I) \cap h(I^{\perp})$. Hence $x \in k(h(I))$ implies that $x \in k(h(I \oplus I^{\perp}))$, and $x \in k(h(I^{\perp}))$ implies that $x \in k(h(I \oplus I^{\perp}))$ so that $k(h(I)) \cup k(h(I^{\perp})) \subset k(h(I \oplus I^{\perp}))$. By the above Corollary 2, $I^{\perp} = k(h(I^{\perp}))$, so that $k(h(I)) \cup I^{\perp} \subset k(h(I \oplus I^{\perp}))$. However, by Theorem 7.4, $k(h(I)) \subset J \cap L^1$, so it follows that $k(h(I)) \cap I^{\perp} = \{0\}$, and thus $k(h(I)) \oplus I \subset k(h(I \oplus I^{\perp}))$. Hence $I \oplus I^{\perp} \subset k(h(I)) \oplus I^{\perp} \subset I \oplus I^{\perp}$, i.e., $I \oplus I^{\perp} = k(h(I)) \oplus I^{\perp}$. Since both members of the last equation are direct sums, it follows that $I = k(h(I))$.

THEOREM 7.6. *If $I \in \mathscr{I}$, then $h(I)^0 = h(J \cap L^1)^0$.*

Proof. Since $I \subset J \cap L^1$, and $\mathrm{cl}\, I = J$, it follows that $\mathrm{cl}\, I = \mathrm{cl}\,(J \cap L^1)$, so that by Corollary 3 of Theorem 7.2, $h(I) = h(J \cap L^1)$ $[m]$. Let $A = h(I)^0 - h(J \cap L^1)$, so that A is open, and thus either $A = \varnothing$ or $m(A) > 0$. But $A \subset h(I) - h(J \cap L^1)$ so that $m(A) = 0$. Hence $A = \varnothing$, and therefore $h(I)^0 \subset h(J \cap L^1)$. Thus $h(I)^0 \subset h(J \cap L^1)^0$. However $h(J \cap L^1) \subset h(I)$, so that $h(J \cap L^1)^0 \subset h(I)^0$, and therefore $h(I)^0 = h(J \cap L^1)^0$.

THEOREM 7.7. *Let $I \in \mathscr{I}$ and let F be a closed subset of $h(J \cap L^1)$. If $h(J \cap L^1) = F$ $[m]$, then $h(J \cap L^1) = F$.*

Proof. As before, $h(I) = h(J \cap L^1)$ $[m]$. Hence, by assumption, $F = h(I)$ $[m]$. Therefore if $x \in k(F)$, $h(I) \subset h(x)$ $[m]$, so that, by Theorem 7.2, Corollary 1, $x \in J \cap L^1$; i.e., $k(F) \subset J \cap L^1$. But if F is closed, then $F = h(k(F)) \supset h(J \cap L^1) \supset F$. Therefore $F = h(J \cap L^1)$.

COROLLARY. *If $\hat{G}$ is not discrete, and if $I \in \mathscr{I}$, then $h(J \cap L^1)$ is perfect.*

Proof. Since $h(J \cap L^1)$ is closed, it can be written $h(J \cap L^1) = S \cup P$ where S is scattered, P is perfect (Sierpiński [16, Chapter 1]). By a theorem of Rudin [12, Theorem 5, p. 41] it follows that $m(S) = 0$ so that $m(h(J \cap L^1) - P) = m(S) = 0$. Hence $P = h(J \cap L^1)$.

With suitable modifications, many of the results of §§4–7 that depend upon th restrictive assumption that G is metric can be extended to the more general case where the metric hypothesis is removed. A consequence of this is that a significant generalization of a theorem due to I.E. Segal will be presented in a forthcoming paper.

Added in proof. I wish to thank Professor David M. Burton for bringing to my attention: S. Kantorovitz, *The annihilator of a closed ideal in a function algebra*. Bull. Res. Council Israel Sect. 9F (1960), 132–134. These results are closely associated with those in §5.

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